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GEOMETRICAL MULTIPLICATION OF SURFACES.

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Among the expressions which have to be transformed to a new system of coordinates in Analytical Mechanics there is one of frequent occurrence, namely the one of the form

$$(y_1z_2 - y_2z_1)(y_4z_3 - y_3z_4) + (z_1x_2 - z_2x_1)(z_4x_3 - z_3x_4) + (x_1y_2 - y_1x_2)(y_3x_4 - x_3y_4) \quad (1)$$

where x_i, y_i, z_i are the rectangular coordinates of a point i. I will show that this expression is *invariant* in regard to a rotation of the system of axes of coordinates.

I will call the expression $S_1S_2 \cos(\widehat{S_1S_2})$ the *geometrical product* of the two plane surfaces S_1 and S_2, analogously to the expression "geometrical product" of two straight lines l_1 and l_2 for $l_1l_2 \cos(\widehat{l_1l_2})$. Let, then, first Δ_1 and Δ_2 be the surfaces of two triangles. We shall have the following

Theorem: The geometrical product of two triangles is equal to the sum of the products of their projections on the planes of coordinates; i. e.

$$\Delta_1\Delta_2 \cos(\Delta_1\Delta_2) = \Delta_{1x}\Delta_{2x} + \Delta_{1y}\Delta_{2y} + \Delta_{1z}\Delta_{2z}\,. \quad (2)$$

It is obvious, that a translation of the system parallel to itself does not alter formula (2); we may therefore suppose, that the two triangles have one vertex in common and that this point is taken as the origin 0 of the axes of coordinates. Let then Δ_1 be formed by the points 0, 1, 2; and Δ_2 by the points 0, 3, 4. We shall have:

$$2\Delta_{1x} = y_1z_2 - z_1y_2\,, \qquad 2\Delta_{2x} = y_4z_3 - y_3z_4\,,$$

$$2\Delta_{1y} = z_1x_2 - x_1z_2\,, \qquad 2\Delta_{2y} = z_4x_3 - z_3x_4\,,$$

$$2\Delta_{1z} = x_1y_2 - y_1x_2\,, \qquad 2\Delta_{2z} = x_4y_3 - x_3y_4\,,$$

by means of which formulæ (1) can be transformed into

$$4(\Delta_{1x}\Delta_{2x} + \Delta_{1y}\Delta_{2y} + \Delta_{1z}\Delta_{2z})\,. \quad (3)$$

On the other hand (1) can be transformed into

$$(x_2x_3 + y_2y_3 + z_2z_3)(x_1x_4 + y_1y_4 + z_1z_4) - (x_2x_4 + y_2y_4 + z_2z_4)(x_3x_1 + y_3y_1 + z_3z_1)$$

by a well known formula due to Euler, or into this equivalent form,

$$r_1r_2r_3r_4\left[\cos(\widehat{14})\cos(\widehat{23}) - \cos(\widehat{24})\cos(\widehat{13})\right], \quad (4)$$

where r_i is the distance of the point i from the origin; $\cos(\widehat{ij})$ is the cos of the angle formed by the lines $0i$ and $0j$. We have, therefore,

$$4(\Delta_{1x}\Delta_{2x}+\Delta_{1y}\Delta_{2y}+\Delta_{1z}\Delta_{2z}) = r_1r_2r_3r_4\left[\cos(\widehat{14})\cos(\widehat{23})-\cos(\widehat{24})\cos(\widehat{13})\right]. \qquad (5)$$

Let us now draw a sphere with the radius unity and the centre in 0. Let, further, (1), (2), (3), (4) be the points of intersection of this sphere with the lines 01, 02, 03, 04. Finally, let (5) be the point of intersection of the great circles $\overset{\frown}{12}$ and $\overset{\frown}{34}$, and A be the angle between them, i. e. $A = \angle\, \widehat{\Delta_1\Delta_2}$. This angle is perfectly determined, if we agree to call positive normals such that a rotation about them will carry a certain point i towards a certain point j, as well in the plane of the triangles as in their projections. Then we shall have the following relations:

$$\cos(\widehat{31}) = \cos(\widehat{35})\cos(\widehat{21}+\widehat{25}) + \sin(\widehat{35})\sin(\widehat{21}+\widehat{25})\cos A\,,$$

$$\cos(\widehat{24}) = \cos(\widehat{25})\cos(\widehat{34}+\widehat{35}) + \sin(\widehat{25})\sin(\widehat{34}+\widehat{35})\cos A\,,$$

$$\cos(\widehat{14}) = \cos(\widehat{21}+\widehat{25})\cos(\widehat{34}+\widehat{35}) + \sin(\widehat{21}+\widehat{25})\sin(\widehat{34}+\widehat{35})\cos A\,,$$

$$\cos(\widehat{23}) = \cos(\widehat{25})\cos(\widehat{35}) + \sin(\widehat{25})\sin(\widehat{35})\cos A\,;$$

from which we find that

$$\cos(\widehat{14})\cos(\widehat{23}) - \cos(\widehat{13})\cos(\widehat{24}) = \sin(\widehat{12})\sin(\widehat{34})\cos A\,.$$

Hence

$$4(\Delta_{1x}\Delta_{2x}+\Delta_{1y}\Delta_{2y}+\Delta_{1z}\Delta_{2z}) = r_1r_2r_3r_4\sin(\widehat{12})\sin(\widehat{34})\cos A\,;$$

or as we have

$$2\Delta_1 = r_1r_2\sin(\widehat{12})\,,\quad 2\Delta_2 = r_3r_4\sin(\widehat{34})\,,$$

we finally obtain

$$\Delta_{1x}\Delta_{2x}+\Delta_{1y}\Delta_{2y}+\Delta_{1z}\Delta_{2z} = \Delta_1\Delta_2\cos(\widehat{\Delta_1\Delta_2})$$

i. e. formula (2), which proves the theorem. It follows from the above, that a translation and rotation of the system of rectangular axes of coordinates into a new one (ξ, η, ζ) with the origin at the point (a, b, c) will change the expression (1) into the following:

$$\begin{aligned}&[(\eta_1+b)(\zeta_2+c)-(\eta_2+b)(\zeta_1+c)]\,[(\eta_4+b)(\zeta_3+c)-(\eta_3+b)(\zeta_4+c)]\\ +\;&[(\zeta_1+c)(\xi_2+a)-(\zeta_2+c)(\xi_1+a)]\,[(\zeta_4+c)(\xi_3+a)-(\zeta_3+c)(\xi_4+a)]\\ +\;&[(\xi_1+a)(\eta_2+b)-(\xi_2+a)(\eta_1+b)]\,[(\xi_4+a)(\eta_3+b)-(\xi_3+a)(\eta_4+b)].\end{aligned}$$

It is well to notice, that we are not restricted to lines purely, because the theorem is true whatever quantities are represented by the lines; thus, for instance, they may represent forces.

The above theorem can be generalized without further details, if the fact that any plane surface can be regarded as the limit of a polygonal surface, be taken into consideration; because every polygon may be divided into triangles, and then the above theorem applied. Hence:

The geometrical product of two plane surfaces S_1 and S_2 is equal to the sum of the products of their projections on the planes of coordinates, i. e.

$$S_1S_2\cos(\widehat{S_1S_2}) = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}\,.$$

To conclude, I will give formula (2) in form of determinants:

$$4\Delta_1\Delta_2\cos(\widehat{\Delta_1\Delta_2}) = \left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{matrix}\right\|$$

$$= \begin{vmatrix} x_1x_4 + y_1y_4 + z_1z_4, & x_1x_3 + y_1y_3 + z_1z_3 \\ x_2x_4 + y_2y_4 + z_2z_4, & x_2x_3 + y_2y_3 + z_2z_3 \end{vmatrix} = -\begin{vmatrix} x_1 & x_2 & -1 & 0 & 0 \\ y_1 & y_2 & 0 & -1 & 0 \\ z_1 & z_2 & 0 & 0 & -1 \\ 0 & 0 & x_3 & y_3 & z_3 \\ 0 & 0 & x_4 & y_4 & z_4 \end{vmatrix}.$$

In the special case, when the triangles are parallel and equal, this formula becomes:

$$4\Delta^2 = \left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix}\right\|^2 \quad \text{or} \quad 2\Delta = \left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix}\right\|$$

a well known formula.

JOHNS HOPKINS UNIVERSITY, *October* 30, 1894.